Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE

SEWANEE REVIEW

VOL. XIV.] JANUARY, 1906. [NO. 1.

SOME PHASES OF MODERN MATHEMATICS

The study of mathematics needs no defense. As some one has said, "every one must count and measure or perish." This sentiment may, however, seem a little irrelevant in speaking of higher mathematics, with which we are chiefly concerned in this article. I have not time or space here to refer to the recognized utility of mathematics as a mental drill or as the useful handmaiden of the arts and sciences. "I hate mathematics" is not an uncommon saying, but surely only the ignoramus would seriously venture to add "it is of no use." The empirical and observational sciences all come to mathematics for help and all bow before it as a science superior to themselves; and the pure mathematician is only too glad to have an opportunity of lending aid to engineering, astronomy, physics, chemistry and other sciences, and on the other hand he realizes that he gets much inspiration and much material aid from these sciences.

But while the larger and more valuable fruits of pure mathematics are to be found in applied mathematics, we can not overlook the fact that, though such abstract reasoning seems dry and unprofitable to some minds, the study of pure mathematics, even outside of its sphere as a mental exercise of the highest order, is uplifting in itself and opens up visions of the true and the beautiful in a way that no inexact science can do. While the writer has in mind, in this paper, these sweeter and better flavored fruits of pure mathematics, he does not mean in the least to disparage applied mathematics, for which he has the greatest respect and admiration. The former is really the mathematics of

precision, the latter, the mathematics of approximation, but the gulf between them is not a broad one.

I think I may safely say that very few people, even among the most cultivated and best educated, have the faintest idea what the great mathematicians of the day are doing, and in what channels mathematical thought has directed itself during the past hundred years. The very names of the world's best mathematical workers are scarcely known outside of a very limited circle of scientists. Now there are some facts concerning the development of mathematics in recent times that can perhaps, even in the limited time at my disposal, be amplified enough to be of some interest to the general reader. Although I propose to write of modern mathematics, let us go back for a moment (by so doing we shall really be sounding the key-note of modern thought) to first principles by asking the question: *What is mathematics?*

For curiosity, if for no better reason, I turn to the "Century Dictionary," and find its definition to be "mathematics is the science of quantity: the study of ideal constructions (often applicable to real problems) and the discovery thereby of relations between the parts of these constructions before unknown." Owing to necessary limitations of space in a work such as the "Century Dictionary," perhaps this definition could not be greatly improved upon; but it is nevertheless entirely inadequate when we call to mind such subjects as projective geometry, the theory of groups, and many other phases of recent mathematical development. Not to do injustice to this dictionary, I ought to add that further light is thrown upon the subject by quotations from Clifford, and there is also given the celebrated definition of Benjamin Peirce: "Mathematics is the science which draws necessary conclusions."

That keen thinker, Professor Simon Newcomb,[1] defines mathematics as "the science which reasons about the relations of magnitudes and numbers, considered simply as quantities admitting of increase, decrease and comparison." Professor Chrystal,[2] the well-known English mathematician, suggests in part the

[1] In Johnson's "Universal Cyclopædia."

[2] Encyclopædia Britannica, Vol. XV.

following definition: "Any conception which is definitely and completely determined by means of a finite number of specifications, say by assigning a finite number of elements, is a mathematical conception." "A triangle," for example, being defined by three elements (a finite number) is a mathematical conception; "a man" is a *non*-mathematical conception, for no finite number of elements is sufficient for an adequate definition.

Now these scholars had no intention of laying down dogmatically a precise definition. No mathematician was ever satisfied with a definition of mathematics. Professor Bôcher, of Harvard, at the St. Louis Congress of Arts and Sciences, delivered an address, on "Fundamental Conceptions and Methods of Mathematics," in which he discusses at some length possible answers to the question: What is mathematics? Among other things, he says: "in order to reach a satisfactory conclusion as to what really characterizes mathematics, one of two methods is open to us. On the one hand we may seek some hidden resemblance in the various objects of mathematical investigation, and having found an aspect common to them all we may fix on this as the one true object of mathematical study. Or, on the other hand, we may abandon the attempt to characterize mathematics by means of its *objects of study*, and seek in its *methods* its distinguishing characteristics. Finally there is the possibility of our combining the two points of view. The first of these methods is that of Kempe, the second will lead us to the definition of Benjamin Peirce, while the third has recently been elaborated at great length by Russell. Other mathematicians have naturally followed out more or less consistently the same ideas, but I shall nevertheless take the liberty of using the names Kempe, Peirce and Russell as convenient designations for these three points of view."

For a rather lengthy and able discussion of these three methods the reader is referred to Bôcher's published address,[3] wherein he shows that the three methods of approach to the question lead in the end to results that stand in intimate relation to one another.

---

[3] *Bulletin of the American Mathematical Society*, Vol. XI, No. 3.

Now, in regard to Peirce's definition: *Mathematics is the science which draws necessary conclusions*, I think two questions spring at once to the mind, and, unless these questions can be at least in a measure answered, the definition seems too vague. These questions are: first, what is meant by *necessary* conclusions? and, second, *from what* are these conclusions drawn? I take the word necessary to refer to the only legitimate (true) conclusions that can come from the premises be they true or false, that is, the conclusions must come whether the premises are true or false. This makes necessary the second question — From what are the conclusions drawn? In the following simple example they are undoubtedly drawn from premises that lead to a result not in accordance with our experience or observation.

Two boys, John and Will, at the same instant, start running on a straight road towards C, Will's starting point (B) being $9\frac{1}{2}$ miles nearer C than John's starting point (A). Will's rate is $\frac{1}{10}$ of a mile a minute. John overtakes Will at a point $\frac{1}{2}$ mile from B. What is John's rate?

We may solve this as follows:

$$\begin{aligned} \text{Let } x &= \text{John's rate, miles per minute.} \\ 10 \text{ miles} &= \text{distance run by John.} \\ \tfrac{1}{2} \text{ miles} &= \text{distance run by Will.} \\ \tfrac{10}{x} &= \text{time John is running.} \\ \tfrac{1}{2} \div \tfrac{1}{10} &= \text{time Will is running.} \\ \therefore \tfrac{1}{2} \div \tfrac{1}{10} &= \tfrac{10}{x};\ \therefore 5x = 10,\ x = 2. \end{aligned}$$

Hence John's rate is *two miles per minute*. Now here we have started with certain premises, and have arrrived at *necessary* conclusions. But, bringing to our aid experience and observation, we feel justified in saying that we have reached a result that is anatomically and physiologically absurd, hence we conclude that our premises are faulty. But our mathematics is all right. A conclusion arrived at mathematically will have just as much empirical truth as the premises, no more, no less. So, in a sense, the mathematician is independent of the truth or falsity of a premise. But, on the other hand, it behooves the mathematician to be very careful about his assumptions, when dealing with a problem that bears intimate relations to other

problems using common mathematical conceptions. Peirce's "necessary conclusions" must not be drawn from anything or everything, otherwise our mathematics would consist only of a series of detached propositions in logic.

Where is the *starting point* in a mathematical investigation? In Euclidean Geometry we might say that the starting point is to be found in certain axioms and postulates that are assumed as true. These axioms and postulates come so directly from experience and observation that Geometry has been classed as a natural science rather than a branch of mathematics. In mathematics it is a general principle that nothing must be assumed that can be proved. I cannot, and I would not if I could, enter into philosophical speculations as to the nature of premises; but it seems to me that if we are going to adopt Peirce's definition, the two questions I have asked are pertinent. It is evident that Peirce's definition emphasizes the deductive character of mathematics, but deduction does not constitute even a major part of mathematical truth. Though the hope of coming to any convincing conclusion is slight, I regret that I cannot follow up this subject further; in fact, much of what I have to say must of necessity be suggestive rather than conclusive.

Just one word more as to the nature of mathematics. It is a mistake often met with a few decades ago that mathematics is mainly reasoning, for intuition and imagination come in for a large share of the work.

"Intuitive or self-evident truths are those which are conceived in the mind immediately; that is, which are perfectly conceived by a single process of induction the moment the facts on which they depend are apprehended without the intervention of other ideas."[4] Simple examples are the axioms of geometry, *e. g.* "a whole is equal to the sum of its parts." I presume that every one feels that he knows this, and would say that it is a self-evident truth. But the truth of some so-called axioms has been questioned, showing that what is apparently self-evident to one mind may not seem so to another. We need no stronger evidence than this to show us that intuition must be used with cau-

---

[4] Davies: "Logic and Utility of Mathematics."

tion. It is not peculiar to Geometry, but is used tacitly, if not avowedly, in all branches of mathematics. To the student of the differential calculus, it seems intuitive that every continuous function must have a derivative, that is, that every continuous curve has tangents; but not very many years ago Weierstrass produced a continuous function without derivatives. This discovery came as a shock to the mathematical world. I give it as an example of the fatal weakness of intuition; and, yet, where would we be without intuition?

It is well known, of course, that induction is frequently employed in mathematics. Induction is at once a strong and weak weapon; very powerful in the hands of the skilful man, weak and uncertain when employed by the rash or ignorant. There are numerous instances in mathematics where important principles have been discovered by induction and later, perhaps years afterwards, proved by vigorous deduction.

As great a man as Huxley made the allegation that "Mathematics is that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing of causation." The history of modern mathematical thought refutes this charge. Sylvester, in an address before the British Association in 1869, gave a powerful answer to this sweeping assertion of Mr. Huxley. In part Sylvester said: "Most, if not all, of the great ideas of modern mathematics have had their authority in observation. Lagrange, than whom no greater authority could be quoted, has expressed emphatically his belief in the importance to the mathematician of the faculty of observation; Gauss called mathematics a science of the eye . . . ; the ever to be lamented Riemann has written a thesis to show that the basis of our conception of space is purely empirical, and our knowledge of its laws the result of observation, that other kinds of space might be conceived to exist subject to laws different from those which govern the actual space in which we are immersed." So spoke Sylvester — a quarter of a century later he could have spoken with even more fervor.

The part played by imagination has perhaps never been sufficiently emphasized, because it is hard to tell exactly where reasoning stops and imagination comes in. It is perhaps this subtle

but strong element that has made some say that mathematics is akin to literature. It was Sylvester, I believe, that said "Mathematics is poetry." Picard[5] said "The idea of number belongs not only to logic, but to history and psychology." Certain it is that in some phases of mathematics we must look outside its pure realms both for a starting point and for material with which to carry on the investigation.

In the opinion of the writer, hair-splitting theories of philosophy have but little place in mathematics, and mathematics would lose its conservative character if they had.

Professor Schubert[6] in his "Essay on the Nature of Mathematical Knowledge" aptly says: "The intrinsic character of mathematical research and knowledge is based essentially on three properties: first, on its conservative attitude towards the old truths and discoveries of mathematics; secondly, on its progressive mode of development due to the incessant acquisition of new knowledge on the basis of the old; and, thirdly, on its self-sufficiency and its consequent absolute independence."

I do not claim, however, that mathematics is the exact science, though the most exact of sciences, that the average person imagines it to be. The foundations of mathematics have not always stood the test of stability, though the superstructure has never been in danger of tottering over. Later on we shall see how, during the nineteenth century, these foundations were examined into and materially strengthened. Now all of this is preliminary to the main subject of my paper; and it may be asked why has the writer gone into these questions at all, suggesting difficulties that may not have been apparent, asking questions, and yet not answering them. As one of the main features of modern mathematics is the going back to first principles, my course is perhaps vindicated. It may be that what has been said so briefly and so imperfectly touching fundamental notions will in a measure prepare our minds for some of the wonderful onslaughts of nineteenth century mathematics.

The discovery of the Calculus by Newton and Leibnitz near the

---

[5] See translation of his St. Louis Address in Vol. XI, No. 8, of *Bulletin of American Mathematical Society*.

[6] *Mathematical Essays and Recreations*, translated by T. J. McCormack.

end of the seventeenth century, opened up a new and vast field for mathematical research. The most powerful tool yet devised had been put into the hands of mathematical students. After its queer notation had become familiar, and the principles and details of operations were fairly well understood, calculus became popular and mathematics made great strides.

France and Switzerland took the lead in the eighteenth century in the development of mathematics. A mere mention of the great names of that period is inspiring: — the Bernouillis, Euler, Lagrange, Laplace, Legendre, Fourier, Monge. Towards the close of the eighteenth and the beginning of the nineteenth centuries, other countries came to the front, while the French and Swiss continued their magnificent work.[7]

Germany had her Gauss, Jacobi, Dirichlet, and more recently such men as Steiner, von Staudt, Plücher, Clebsch, Felix Klein, Weierstrass, Riemann, Fuchs, etc.; England produced DeMorgan, Boole, Hamilton, and more recently Cayley and Sylvester. Russia entered the list with Lobatchewsky; Norway with a mathematical giant, Abel; Italy, with Cremona; Hungary, with her two Bolyais, the United States with Benjamin Peirce, while France, still well to the front, produced Cauchy, Galois, Poncelet, Chasles, and others; and among those now living, Picard, Darboux and Poincaré. These nineteenth century men, as Newton said of himself, have stood on the shoulders of giants, but they themselves are not pigmies.

It has been well said that the chief characteristic of modern mathematics is its generalizing tendency. The great propositions have been general ones, some special cases of which have often been found interesting either in themselves or in an application to some practical problem.

Mathematics may be considered under three general heads: — Arithmetic (number), Geometry (form), Analysis (function). It would be more scientific to treat my subject under these three heads; but I have found it impossible to adhere strictly to this formal division, and, at best, I can merely mention a few things done

[7] In this general resumé the writer does not wish to draw invidious comparisons, and he is entirely conscious that some names may be omitted that stand higher in the mathematical world than some that are mentioned.

in these three fields, avoiding as far as possible dates and technical terms. It must be remembered that only a few mathematicians can be mentioned in a paper of this length, and many phases of mathematics must be passed by in silence. So various are the departments of modern mathematics that the student can make a specialty of only a few of them, and some of them he will barely know by name. It is far beyond the intention or scope of this paper to point out the practical application of many an abstract problem of modern mathematics. Applied mathematics has often been enriched in the most surprising manner by transcendental investigations.

ARITHMETIC

Arithmetic, or the Theory of Numbers, is the basis of all mathematics. In its elementary forms and its practical applications to commercial life, this branch of our science is commonly known as arithmetic; in its higher forms and especially in its generalizations, Theory of Numbers is the name assigned to it. Number is the keynote of the science of mathematics. Arithmetic suggests counting, calculating. We at once think of the Arabic system of Numerals, Decimal Fractions and Logarithms, those three inventions that facilitated numerical calculations more than all other discoveries in mathematics. The greatest names of old connected with the *theory of numbers* were Fermat, Euler and Lagrange. Later came the great Gauss who revolutionized the theory of numbers. He it was who called mathematics the queen of the sciences and arithmetic the queen of mathematics. Gauss [8] was pronounced by Laplace the greatest mathematician in all Europe. At the age of twenty, he had overturned old theories and old methods in all branches of higher mathematics. He was the first to observe rigor in the treatment of infinite series, the first fully to recognize and emphasize the importance of determinants and imaginaries and to make systematic use of them; he was the first to arrive at the method of least squares, and the first to observe the double periodicity of elliptic func-

[8] Karl Friedrich Gauss, the German mathematician, physicist and astronomer; born 1777, died 1855. The writer has copied this little account of Gaus almost *verbatim* from Cajori's "History of Mathematics."

tions. In the fields of physics and astronomy, he reconstructed the whole of magnetic science, and originated a beautiful method for computing the orbit of a planet from three observations. From 1807 till his death he was director of the observatory at Göttingen. Gauss's *Disquisitiones arithmeticae*, published in 1801, was epoch-making, giving as it did the most important part of the elementary development of the theory of numbers.

Dirichlet, who succeeded Gauss as professor at Göttingen, is said to have been the first to make this great work of Gauss transparent and intelligible. Dirichlet made many important contributions to the theory of numbers. Among other things, he proved that for integral numbers x^5+y^5 can not equal z^5, which is a special case of Fermat's proposition "x^n+y^n can not equal z^n, when n is greater than 2." Euler and Legendre had proved it for $n=3$, and $n=4$. Particularly noteworthy were Dirichlet's applications of analysis to the theory of numbers.

The next great advance in this branch of mathematics was made by Kummer by the invention of his "ideal numbers." This wonderful conception revived the subject of numbers, which was later enriched by Kronecker, Dedekind, Smith, Hilbert and others.

ANALYSIS

Under this head, let us consider for a moment imaginary and complex numbers and the special methods of calculation in which they are used. Not to go minutely into its previous history, the value of the so-called imaginary unit was hardly recognized before about the beginning of the nineteenth century, though it was not the despised creature that it was when first given the somewhat opprobious title of "imaginary" or "unreal." It is said that Kuhn, in 1750 or '51, first gave geometric expression to $\sqrt{-1}$, and it was not till 1797 that Wessel (of Norway) published the first clear, accurate and scientific treatment of lines represented by quantities of the form $a+b\sqrt{-1}$.[9] The credit of this is usually attributed to Argand, who in 1806 published his often referred to "Essai." [10] Now what finally grew out of

[9] See Address by W. W. Beman, before Section A of the A. A. A. S., August, 1897.

[10] *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques.*

this? Nearly all modern theories employ the complex number, and many owe their very existence to it.

Among the latter, we have *Quaternions*, invented by Sir Wm. Rowan Hamilton in 1843. In the *North British Review* for Sept. 1866, there is an interesting sketch of Lord Hamilton's life, in which is described his invention of Quaternions. In the midst of his studies in this direction, the key-note of the whole system suddenly came to him on the 16th of October, 1843, while walking with his wife one evening along the Royal Canal in Dublin; and he then and there "pulled out, on the spot, a pocket-book" and wrote down the fundamental formulae $i^2 = j^2 = k^2 = ijk = -1$. In a word, by removing the restriction that the commutative law holds in multiplication (that is $ab = ba$), Hamilton was able to build up his new system. To give any idea what this new system was, would require too much space. The solution of many and varied problems is very much shortened by the employment of quaternions. The method is of great use in certain problems in physics; but its value in this field is hardly as great as it was originally claimed it would be.

In 1844, Hermann Grassman, published the first part of his wonderful work, called *Ausdehnungslehre*, which may be translated "The Science of Extensive Quantities," or "Directional Calculus." There are many that think that Grassman's calculus has points of superiority over Hamilton's quaternions, which it resembles. Grassman's method is not confined to any particular dimension, and is said to be especially convenient in dealing with n-dimensional problems. Another elegant method, in which the imaginary number plays the chief role, is known as the *Method of Equipollences*, and was invented by the Italian mathematician, Bellavitis, a few years before Hamilton's Quaternions appeared. *Equipollences*, however, are restricted to a plane, while quaternions hold in space.

Benjamin Peirce, who was professor of mathematics at Harvard from 1833 till his death in 1880, made profound researches in *Linear Associative Algebra*. His method, unlike that of Hamilton and Grassman, was not geometric. He still holds the palm, I believe, of being the most able mathematician that America has ever produced — it is certain that his algebraic

work was epoch-making. Among Americans of note, to make a little digression, might be mentioned George William Hill and Professor Simon Newcomb, though their special work has been in Astronomy rather than pure mathematics. Concerning the former (G. W. Hill), Poincaré[11] writes: . . . *mais son œuvre propre, celle qui fera son nom immortel, c'est sa théorie de Lune; c'est là qu'il a été non seulement un artiste habile, un chercheur curieux, mais un inventeur original et profond.* To say nothing of the noted teachers in our greater universities, the American Mathematical Society is doing much to stimulate our younger mathematicians, among whom there is at present so much activity that the writer feels emboldened to predict that the present century will witness some native American stars in the mathematical galaxy equal to those that are conspicuous in European nations.

Perhaps the most studied, the most remarkable and the most fascinating branch of nineteenth century mathematics is the "Theory of Functions." Nearly all the famous mathematicians of this period have done something to advance this many-sided theory. For the real beginnings of the theory of functions,[12] especially that of the elliptic and Abelian functions, we must look back to Faguano, Maclaurin, D'Alembert, and Landen; but I cannot pause long enough to consider these beginnings, or to follow the history of its development through Euler, Lagrange, and Legendre. Suffice it to say that the general theory fairly launched at the begining of the nineteenth century by Legendre, Jacobi and Abel, was carried on by such men as Gauss, Cauchy, Dirichlet, Riemann, Weierstrass and others. In an article such as this, intended as it is for the eye of the layman rather than the mathematician, it would be rather absurd to pretend to convey any adequate idea of what the Theory of Functions really is. It would be well, however, at the outset, to give a general definition of what is meant by a *function*. $F(x)$ [to use a common notation] is a function of x throughout an inter-

---

[11] In Preface to "The Collected Mathematical Works of George William Hill," Vol. I, published by the Carnegie Institution of Washington, June, 1905.

[12] Beman and Smith's translation of Dr. Karl Fink's *Geschichte der Elementar-Mathematik*, Open Court Publishing Co.

val, when, to every value of x within the interval, belongs one or more definite values of $F(x)$. This definition, while it does not cover the whole ground [13] will serve to fix our ideas. In the usual college curricula such functions are first met with in trigonometry and analytic geometry. We have the "theory of functions of the real variable" and "the theory of functions of the complex variable," though they are often studied side by side. Riemann defines a function of a complex variable as follows: "A variable complex quantity w is called a function of another variable complex quantity z $(z=x+iy)$, if w change with z in conformity to the equation $\frac{\delta w}{\delta y}=i\frac{\delta w}{\delta x}$."

From which we see that a function of a complex variable is a function that contains x and y in the definite combination $x+iy$. As I have just intimated, what the theory of functions consists of, is a long story; but I might say that in this branch of mathematics all sorts of simple and complicated functions are considered, and the great point is to find how such functions behave, what their singular points are, etc., *within certain domains*. In connection with the domain or region in which the function holds sway, or possesses certain properties, many interesting things are often discovered. Series play an important part in the development of the Theory of Functions, and one remarkable thing is that divergent series, that invention of the devil as Abel said, are assuming importance.

Elliptic Functions are an interesting class of functions. The history of functions as first developed is largely a theory of algebraic functions and their integrals. Cauchy, Riemann and Weierstrass worked along distinct lines, but the ideas of the three were united before the close of the century, and there is now really only one theory of functions.[14] Besides those already mentioned, among the names especially associated with the The-

---

[13] It would lead us too far afield to enter into any discussion of continuous and discontinuous functions, monogenic and non-monogenic functions, etc.

[14] Here, and in some other places I have borrowed from an address delivered by Professor James Pierpont before the St. Louis Congress of Arts and Sciences. This most interesting address, entitled "The History of Mathematics in the Nineteenth Century," was published in the *Bulletin of the American Mathematical Society*, *Vol. XI*, *No. 3*.

ory of Functions, are Mittag-Leffler, Poincaré, Hadamard, Laguerre, Clebsch, Swartz and Klein.

Another fruitful field of mathematical activity, and one that owes much to the theory of functions, is that of Differential Equations. Here the names of Cauchy, Fuchs, Poincaré, Gauss, Kummer, Riemann and Lie are prominent. In this field Poincaré found a divergent series useful, and of importance is the use by others of the celebrated hypergeometric equation.

Next to the theory of functions, the dominant idea of the century was the *group concept*. The Group Theory gets its name from the fact that every algebraic equation has attached to it a certain group of substitutions. Galois may be considered the founder of the modern theory of Groups, which is especially useful in the development of Differential Equations. Lie gave the latter subject new life by the employment of groups. Klein was especially prominent in these investigations, and the names of Cayley and Sylvester appear here as in so many other places.

There are other special theories of analysis too numerous even to mention by name. To satisfy the needs of all these modern theories, it became necessary to coin many words, or to endow old terms with a special local meaning. For example, the terms *invariant*, *discriminant*, *Hessian*, *Jacobian*, are due to Sylvester, who introduced so many new names that he was playfully called the mathematical Adam.

Before leaving the general subject of analysis, I should like to say just a word about a mathematician whose name has been several times mentioned, — I refer to the Norwegian, Neils Hendrick Abel (1802–29) who, with the possible exception of Gauss, had the greatest mathematical mind of the nineteenth century. He died in his twenty-seventh year, and yet in this short span of life "he penetrated new fields of research, the development of which has kept mathematicians busy for over half a century." [15] He made important discoveries in elliptic functions, but his greatest work lay in his profound investigations in what are now known as Abelian Functions. Jacobi pronounced this theorem as the greatest discovery of the century on the integral calculus. One

---

[15] "History of Mathematics," by Florian Cajori.

of his earlier achievements was the proof that a general solution by radicals of an equation of the fifth or higher degrees, was impossible. Abel[16] is represented as being over-sensitive and very modest. This stood in the way of the recognition of his worth, and it was not till many years after his death that his transcendent genius was appreciated. He was hampered also by poverty, having to give private lessons to eke out an existence. His poverty and ill-health made his life a pathetic one. Finally Crelle, the founder and editor of the great journal, secured an appointment for him at Berlin, but the news of it did not reach Norway till after the death of Abel.

GEOMETRY

Now, just for a few moments, let us lastly hastily glance over the geometrical field. In Geometry, the discoveries and developments made during the nineteenth century were just as wonderful and startling as those made in analysis. We often find the same mathematicians working along both lines, and, of course, the line of demarcation between these two great branches of the science is not sharply drawn. Neither could well get along without the other, — in this respect, instead of considering them as rivals, we might compare them to husband and wife, though which is the husband and which is the wife it would be hard to say. Of the offspring, some incline to Geometry, others to analysis.

Now, in speaking of Geometry, I do not refer to that ancient ancestor, Euclid. He would probably not recognize Modern Geometry as anything closer than a collateral descendant, and the bare mention of hyperbolic or elliptic geometry would, I fear, make the old geometer turn over in his grave.

But, to be serious, the father of modern geometry was Monge (1746–1818), who we may say created descriptive geometry, thus doing much for engineering. The great exponents of what is known as *Modern Geometry* were Chasles, of Paris, and Steiner, of Berlin. Both devoted their lives to pure geometry, preferring synthesis to analysis, and proposing to institute a rival to carte-

---

[16] Prof. Bjerknes wrote a life of Abel, a French translation of which was published in 1885.

sian analysis. A little later Von Staudt brought out his great work on Geometry, *Geometrie der Lage*, followed a few years later by his *Beiträge zur Geometrie der Lage*, and Cremona in Italy published his elegant work on "Projective Geometry." All these and others, too, published many valuable memoirs and papers in the journals of the day. I cannot here point out even the main points of difference in the methods of these geometers. And time will not permit us to consider the many and varied recondite problems of geometry that are constantly coming up in deep researches. The eminent French mathematician, Darboux, read at the St. Louis Congress a superb paper on "A Survey of the Development of Geometric Methods." A translation of it appeared in the *Popular Science Monthly* and another in the *Bulletin of the American Mathematical Society*. It was interesting that Picard spoke on analysis at that same meeting. It was fortunate that the three greatest French mathematicians were present and spoke at that Congress — Darboux, Picard and Poincaré, the last perhaps the greatest living mathematician in the world.

I have devoted so little time to these great geometries and geometers that it would hardly be consistent to say much, if anything, about the so-called non-Euclidean geometry. Dr. George Bruce Halsted, formerly of the University of Texas, now of Kenyon College, who is the great disciple of this interesting subject in America, has translated the works of Lobachewsky and Bolyai, the pioneers in the non-euclidean field.

Euclid assumed as axiomatic, that, "if two lines are cut by a third, and the sum of the interior angles on the same side of the cutting line is less than two right angles, the lines will meet on that side when sufficiently produced." This is the so-called vicious parallel axiom, of which so much has been written. After centuries of fruitless attempts to prove this assumption, the bold idea dawned upon the minds of several mathematicians that a geometry might be built up without assuming the parallel axiom; and Lobachewsky and Bolyai brought out publications which actually assumed the contradictory of that axiom.

The fundamental proposition of the new geometry is: The sum of the angles of a triangle at least in any restricted portion

of the plane, is *equal to, less than,* or *greater than* two right angles. This gives *three* hypotheses, which are exhibited also in the following theorem: The angles at the extremities of two equal perpendiculars are either right angles, acute angles, or obtuse angles, at least for restricted figures.

We may speak of the three cases as respectively: the hypothesis of the right angle, the hypothesis of the acute angle, the hypothesis of the obtuse angle, and these three hypotheses give rise, respectively, to

The Parabolic Geometry — Euclid;
The Hyperbolic Geometry — Lobachewsky;
The Elliptic Geometry — Riemann.

If you have followed me closely, you will see that Euclidean Geometry, which we all love, is not destroyed by the new ideas, but is shown to be a special case of a more general system. It seems to me that we live and move and have our being in Euclidean space. It may be that in some far off region — in some other universe if you will — there are people existing in hyperbolic and elliptic space, where the spirit of Lobachewsky and Riemann hold sway.

THE CRITICAL MOVEMENT

The Study of Functions of Real Variables gave birth to the critical movement which began about the beginning of the century and culminated, we may say, in Weierstrass. This critical spirit, as I have before intimated, is one of the salient characteristics of modern mathematical thought. It began with Lagrange and Gauss, then Cauchy introduced rigor into calculus, and Abel, Bolzano, Dirichlet and others continued to question assumptions and to examine critically into the foundations of the science. But the man who stands head and shoulders above all others in this movement is the greatest of modern German mathematicians, Weierstrass.[17] He started at the very foundations of arithmetic and geometry and after strengthening those foundations, he began slowly and carefully to build upon them, clearing up many obscurities as he went. On a purely arithmetical basis, with no appeal to our intuition, Weierstrass developed his splen-

[17] Karl Theodor Wilhelm Weierstrass (1815–1897).

did theory of functions. The effect of this critical spirit was soon felt all over the world. Dedekind and Cantor have introduced this rigor into the Theory of Numbers, while Harnack, Jordan, Vallée-Poussin and others have performed a similar service for calculus. Noteworthy in this connection are the critical investigations of recent Italian mathematicians under the leadership of Peano. Growing out of this movement, we can see, I think, among students of calculus a growing spirit of reverence and admiration for Cauchy.

Now, some one may ask: "What is this to me? Weierstrass' transcendent theories may be very beautiful, his profound criticism may be stimulating to the specialist in mathematics, but, even if I knew something about these matters, they would doubtless profit me not at all." In fact, the non-mathematician may reasonably inquire whether the usual undergraduate college courses are in any degree affected by modern mathematical investigations. I answer, emphatically, *yes*.

Let me close this article by giving one or two exmples of what the influence of Weierstrass has done, — this man who in his professor's chair at Berlin startled the world by his brilliant work in higher mathematics. I purposely take my illustrations from elementary mathematics as taught in America.

In nearly every algebra written or revised within the last five years a definition [18] of multiplication is found which, I think I may safely say, no American algebra contained ten years ago. The definition is this: "to multiply one number by another is to do to the multiplicand what was done to 1 to get the multiplier." This definition does what no previous definition had ever done, — it fully explains [19] the rule of signs: *The product of two numbers having like signs is positive; and the product of two numbers having unlike signs is negative.*

This definition is due to Weierstrass, and our American school boys have been directly benefited by him. Many other instances could be given. It was Weierstrass who pointed out

[18] If the words of this definition are not given, the spirit of the critical movement is manifest in what is given.

[19] See some recent Algebra — "Fischer and Schwatt's Higher Algebra" for example.

that there are only four fundamental mathematical operations, namely: addition, subtraction, multiplication and division.

It is only recently that the writer has succeeded in finding, for use as a text-book, an American calculus that gives a rigorous proof of Taylor's Theorem; and the number that now do so can perhaps be counted on the fingers of one hand. But the ice is broken, and we shall soon find other text-book writers wheeling into line. However, we cannot here enter into the vices and devices of text-books. It is certain that the critical movement is already doing much for them.

SAMUEL M. BARTON.

The University of the South.